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The Cauchy–Poisson problem for a viscous liquid

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The axisymmetric, free-surface response of a semi-infinite viscous liquid to either a point impulse or an initial displacement of zero net volume is calculated. The asymptotic disturbance is resolved into three components: (i) a damped gravity wave, which represents a primary balance between gravitational and inertial forces with secondary, but cumulative, modification by viscous forces; (ii) a diffusive motion, which represents a balance between viscous and inertial forces; (iii) a creep wave, which represents a balance between gravitational and viscous forces. Van Dorn has suggested that the results may be relevant to the concentric circular ridges that surround the crater Orientale on the Moon.

1. Introduction

Van Dorn (1968) has suggested that the concentric circular ridges that surround the crater Orientale at lat. 20° S. and long. 95° W. on the Moon may have been initiated as gravity waves on a viscous liquid under the impact of a meteorite. Testing the plausibility of this conjecture would require not only the solution of the Cauchy–Poisson problem for a viscous liquid, but, as minimum data, estimates of the viscosity of the material and of the ultimate shear stress (above which the material might behave as a liquid); unfortunately, reliable estimates do not appear to be available at this time. Nevertheless, it does appear worth while to obtain a formal solution of the problem and to examine its qualitative features *vis-à-vis* the classical problem.

The transient development of one-dimensional surface waves on a viscous fluid has been considered by Sretenskii (1941), and the basic motions that such a development comprises were discussed by both Basset (1888, §520–2) and Lamb (1932, §349; Lamb concluded with the statement: 'By a proper synthesis of the various normal modes it must be possible to represent the decay of an arbitrary initial disturbance'). These basic motions may be classified into three types: (i) damped gravity waves, which represent a primary balance between gravitational and inertial forces with secondary, but cumulative, modification by viscous forces; (ii) diffusion, which represents a primary balance between viscous and inertial forces; (iii) creep, which represents a primary balance between gravitational and viscous forces. The relative importance of these

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motions for a particular initial configuration depends essentially on the *viscous length*

$$l \equiv g^{-\frac{1}{3}}\nu^{\frac{2}{3}}, \tag{1.1}$$

where g is the acceleration of gravity and ν is the kinematic viscosity.

We consider a free-surface motion that stems either from an initial displacement of vertical scale d and lateral scale a or from an impulse of magnitude I and lateral scale a. We neglect surface tension, T, on the hypothesis that the capillary length is negligible:

$$l' \equiv (T/\rho g)^{\frac{1}{2}} \ll \max\{l, a\}. \tag{1.2}$$

The available similarity parameters then are

$$a/l = \{(ga)^{\frac{1}{2}}a/\nu\}^{\frac{2}{3}} \equiv 2\lambda^{\frac{1}{2}} \equiv R^{\frac{2}{3}}, \tag{1.3}$$

(we find λ and the Reynolds number R convenient in §5 below), d/a, and

$$v/(ga)^{\frac{1}{2}} \equiv (I/\rho a^3)/(ga)^{\frac{1}{2}}, \tag{1.4}$$

where v is an *equivalent* impact speed (we do not assume conservation of momentum in the impact, so that v and I may not be simply related to the speed and momentum of the impacting body).

The classical, Cauchy–Poisson problem for the response of an inviscid liquid to a concentrated ($a \to 0$) impulse or displacement yields an asymptotic represention of the free-surface displacement in the form

$$\zeta(r,t) \sim A(r,t)\cos\omega + B(r,t)\sin\omega, \tag{1.5a}$$

where

$$\omega = gt^2/4r \to \infty, \tag{1.5b}$$

and A and B are slowly varying functions of r and t. Referring to Lamb's (1932, §349) discussion of the basic motions enumerated above, we may incorporate the dominant effect of a sufficiently small viscosity by neglecting motions of types (ii) and (iii) and multiplying the right-hand side of (1.5a) by the damping factor

$$\exp(-2\nu k_s^2 t) \equiv \exp(-\nu g^2 t^5/8r^4), \tag{1.6a}$$

where

$$k_s = gt^2/4r^2 \equiv \omega/r \tag{1.6b}$$

is the wave-number of stationary phase. Having this result, we seek to determine more precisely those conditions under which motions of types (ii) and (iii) are either negligible or significant in comparison with the damped gravity wave and to determine the modification of the damping factor of (1.6a) for an initial displacement of zero net volume.

Our study having been induced by reference to an impact of cataclysmic magnitude, it seems necessary to consider the implications of the hypothesis of small disturbances, without which significant mathematical progress would be impossible. Strictly speaking, this hypothesis is equivalent to the restrictions

$$d \ll \max\{l, a\}, \quad v \ll \max\{(ga)^{\frac{1}{2}}, (gl)^{\frac{1}{2}}\}; \tag{1.7a, b}$$

however, both intuitive considerations and observational data for underwater explosions (Cole 1948) suggest that events at sufficient distances from the point of impact may be adequately described by ignoring the details of the impact (or explosion) and starting from the assumption of an initial cavity, the details of which must be inferred from both observation and empirical considerations. It seems quite unlikely, on the other hand, that the effects of an impact can be adequately represented by the assumption of a prescribed impulse unless (1.7*b*) is actually satisfied. The formal solution for a point impulse is, nevertheless, of interest both as a preliminary example that yields an especially interesting creep wave and for direct comparison with experiments in actual, viscous liquids.

2. Formulation of problem

We consider a liquid that fills the half-space $z > 0$ (z is positive downwards), is at rest for $t < 0$, and is subjected to a free-surface impulse $\rho\phi_0(r)$ and free-surface displacement $\zeta_0(r)$ (positive downwards) at $t = 0$. Letting

$$p(r,z,t) \equiv \rho\phi_t(r,z,t) \tag{2.1}$$

be the pressure (we follow Lamb's convention for the sign of the potential ϕ) and $r\psi(r,z,t)$ be a Stokes stream function, such that the radial and downward components of the particle velocity $\mathbf{q}$ are given by

$$u = -\phi_r + \psi_z, \quad w = -\phi_z - r^{-1}(r\psi)_r, \tag{2.2}$$

and invoking the hypothesis of small disturbances, we find that the continuity and (linearized) Navier–Stokes equations,

$$\nabla . \mathbf{q} = 0, \quad \mathbf{q}_t = -\nabla(p/\rho) + \nu\nabla^2\mathbf{q}, \tag{2.3}$$

imply

$$\phi_{rr} + r^{-1}\phi_r + \phi_{zz} = 0 \tag{2.4a}$$

and

$$\psi_{rr} + r^{-1}\psi_r - r^{-2}\psi + \psi_{zz} = \nu^{-1}\psi_t. \tag{2.4b}$$

The initial conditions are

$$\phi(r,0,0) = \phi_0(r), \quad \zeta(r,0) = \zeta_0(r). \tag{2.5}$$

The total impulse (positive downwards) delivered to the surface at $t = 0$ is given by

$$I = 2\pi\rho \int_0^\infty \phi_0(r)\, r\, dr. \tag{2.6}$$

The potential energy associated with the initial displacement is given by

$$E_0 = \pi\rho g \int_0^\infty \zeta_0^2(r)\, r\, dr, \tag{2.7}$$

where $\zeta_0(r)$ satisfies the constraint

$$\int_0^\infty \zeta_0(r)\, r\, dr = 0 \tag{2.8}$$

if the displaced volume vanishes identically.

Invoking the requirements of kinematic and dynamical equilibrium at the disturbed free surface, $z = \zeta(r,t)$, we obtain the linearized boundary conditions

$$\zeta_t = w, \tag{2.9a}$$

$$\phi_t + g\zeta - 2\nu w_z = 0, \tag{2.9b}$$

and

$$\nu(u_z + w_r) = 0. \tag{2.9c}$$

We may satisfy these boundary conditions of $z = 0$, rather than $z = \zeta$, without introducing an error greater than that already implicit in the linearization of the equations of motion.

3. Formal solution

We obtain a formal solution to the problem posed in the preceding section with the aid of the following Laplace and Hankel transformations:

$$\{Z, \Phi, W\} = \int_0^\infty e^{-st} ds \int_0^\infty \{\zeta, \phi, w\} J_0(kr) \, r \, dr, \tag{3.1a}$$

$$\{\Psi, U\} = \int_0^\infty e^{-st} ds \int_0^\infty \{\psi, u\} J_1(kr) \, r \, dr, \tag{3.1b}$$

and

$$\{Z_0, \Phi_0\} = \int_0^\infty \{\zeta_0, \phi_0\} J_0(kr) \, r \, dr. \tag{3.2}$$

Carrying out the corresponding transformations of (2.2), (2.4) and (2.9), we obtain

$$U = k\Phi + \Psi_z, \quad W = -\Phi_z - k\Psi, \tag{3.3}$$

$$\Phi_{zz} - k^2\Phi = 0, \quad \Psi_{zz} - \kappa^2\Psi = 0, \tag{3.4}$$

$$sZ - W = Z_0 \quad (z = 0), \tag{3.5a}$$

$$s\Phi + gZ - 2\nu W_z = \Phi_0 \quad (z = 0), \tag{3.5b}$$

and

$$U_z - kW = 0 \quad (z = 0), \tag{3.5c}$$

where

$$\kappa = \{k^2 + (s/\nu)\}^{\frac{1}{2}} \quad (\mathscr{R}\kappa \geqslant 0). \tag{3.6}$$

Invoking the requirement that both Φ and Ψ be bounded as $z \to \infty$, we pose the solution to (3.4) in the form

$$\Phi = A(s, k) \, e^{-kz}, \quad \Psi = B(s, k) \, e^{-\kappa z} \quad (z \geqslant 0). \tag{3.7}$$

Substituting (3.7) into (3.3) and (3.5), we obtain

$$U = kA \, e^{-kz} - \kappa B \, e^{-\kappa z}, \quad W = k(A \, e^{-kz} - B \, e^{-\kappa z}), \tag{3.8}$$

and the matrix equation

$$\mathbf{M} . \{Z, A, B\} = \{Z_0, \Phi_0, 0\}, \tag{3.9}$$

where

$$\mathbf{M} = \begin{bmatrix} s & -k & k \\ g & s + 2\nu k^2 & -2\nu k\kappa \\ 0 & -2k^2 & k^2 + \kappa^2 \end{bmatrix}. \tag{3.10}$$

Solving (3.9) for Z, we obtain

$$Z(k, s) = \frac{Z_0(k)}{s} \left\{ 1 - \frac{gk}{D(s, k)} \right\} + \frac{k\Phi_0(\kappa)}{D(s, k)}, \tag{3.11}$$

where

$$D(s, k) = (s + 2\nu k^2)^2 - 4\nu^{\frac{3}{2}} k^3 (s + \nu k^2)^{\frac{1}{2}} + gk. \tag{3.12}$$

We satisfy the requirement that the real part of the radical κ be non-negative by cutting the s-plane along $s = [-\infty, -\nu k^2]$ and choosing the positive square root along $s = (-\nu k^2, \infty)$. We then find that D has two, and only two, zeros in the cut plane, both of which lie in $\mathscr{R}s < 0$ (see §4 below); accordingly, we may choose the path of integration for the inverse Laplace transformation along $\mathscr{R}s = 0$ to obtain the formal solution

$$\zeta(r,t) = \frac{1}{2\pi i}\int_0^\infty J_0(kr)\,k\,dk\int_{-i\infty}^{i\infty} Z(k,s)\,e^{st}\,ds, \tag{3.13}$$

where Z is given by (3.11) and (3.2). We notice that the effects of surface tension could be included simply by multiplying gk by $1+(kl')^2$ in (3.11) and (3.12).

4. Point-impulse problem

We assume, in this section, that the radius of the area over which the impulse I is applied is negligible compared with the viscous length l and that the initial displacement vanishes identically. Invoking these hypotheses, together with (2.6), in (3.2), we obtain

$$Z_0(k) \equiv 0, \quad \Phi_0(k) = I/2\pi\rho \equiv \nu l h, \tag{4.1}$$

where l is defined by (1.1) and h is an appropriate vertical scale. Substituting (4.1) into (3.11) and (3.13) and introducing the dimensionless quantities

$$\eta = r/l, \quad \tau = (g/l)^{\frac{1}{2}}t, \quad \alpha = kl, \quad \sigma = (l/g)^{\frac{1}{2}}s, \tag{4.2}$$

and $$\Delta(\sigma,\alpha) \equiv (l/g)D(s,k) = (\sigma+2\alpha^2)^2 - 4\alpha^3(\sigma+\alpha^2)^{\frac{1}{2}} + \alpha, \tag{4.3}$$

we obtain $$\zeta(r,t) = h\int_0^\infty J_0(\alpha\eta)\,\chi(\tau,\alpha)\,\alpha^2\,d\alpha, \tag{4.4}$$

where $$\chi(\tau,\alpha) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{\sigma\tau}\,d\sigma}{\Delta(\sigma,\alpha)}. \tag{4.5}$$

We seek the asymptotic behaviour of ζ as $\tau\to\infty$.

Deforming the path of integration in the σ-plane in the usual way, we obtain contributions to χ from the two poles determined by the dispersion equation

$$\Delta(\alpha,\sigma) = 0, \quad \sigma = \sigma_\pm(\alpha), \tag{4.6}$$

and from the two sides of the cut along $\sigma = [-\infty, -\alpha^2]$. The zeros of Δ in this cut plane are (cf. Lamb 1932, §349) complex conjugates in $\mathscr{R}\sigma < 0$ for

$$0 < \alpha < \alpha_c = 1{\cdot}20,$$

negative real in $(-\alpha^2, 0)$ for $\alpha > \alpha_c$, and analytic functions of α in a complex-α plane cut along $\alpha = [0, \alpha_c]$. Their numerical values are plotted in figure 1. Invoking Cauchy's residue theorem to obtain the contributions of the poles to χ and introducing the change of variable

$$\sigma = -\alpha^2(1+x^2) \pm i0, \quad (\sigma+\alpha^2)^{\frac{1}{2}} = \pm i\alpha x$$

on the upper and lower sides of the cut, we obtain

$$\chi(\tau,\alpha) = \sum_{\sigma=\sigma_\pm}\frac{e^{\sigma\tau}}{(\partial\Delta/\partial\sigma)} - \frac{8\alpha^4}{\pi}\int_0^\infty\frac{\exp\{-(1+x^2)\alpha^2\tau\}x^2\,dx}{\{1+\alpha^3(1-x^2)^2\}^2+16\alpha^6x^2}. \tag{4.7}$$

The dominant contributions to the integral (4.4) as $\tau \to \infty$ must come from the neighbourhoods of $\alpha = 0$ and $\alpha = \infty$ since the integrand is exponentially small at intermediate values of α.† The limiting forms of $\sigma_{\pm}(\alpha)$, as determined by (4.3) and (4.6), are given by

$$\sigma_{\pm} = \pm i\alpha^{\frac{1}{2}} - 2\alpha^2 + O(\alpha^{\frac{11}{4}}) \quad (\alpha \to 0), \tag{4.8a}$$

$$\sigma_{+} \sim -\tfrac{1}{2}\alpha^{-1} + O(\alpha^{-4}), \quad \sigma_{-} \sim -0{\cdot}93\alpha^2 + O(\alpha^{-1}) \quad (\alpha \to \infty). \tag{4.8b, c}$$

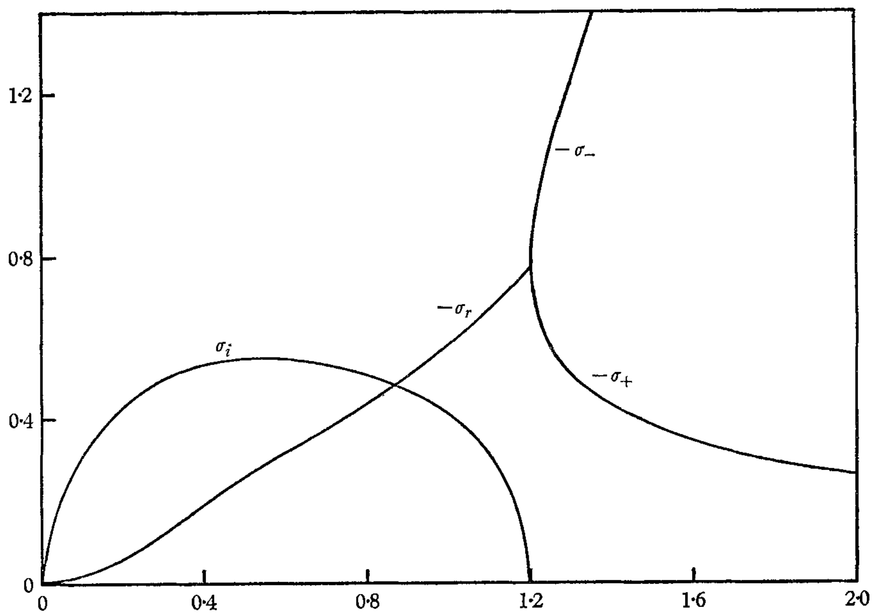

FIGURE 1. The roots of the dispersion equation (4.6); $\sigma = \sigma_r \pm i\sigma_i$ for $0 < \alpha < 1{\cdot}20$.

Substituting these approximations and the corresponding approximations to $\partial\Delta/\partial\sigma$, as determined from (4.3), into (4.7) and approximating the x-integral with the aid of Watson's lemma, we obtain

$$\chi \sim \alpha^{-\frac{1}{2}} \exp\{-2\alpha^2\tau\} \sin(\alpha^{\frac{1}{2}}\tau)\{1 + O(\alpha^{\frac{9}{4}})\} - 2(\pi\tau^3)^{-\frac{1}{2}} \alpha \exp\{-\alpha^2\tau\}\{1 + O(\alpha/\tau)\}$$
$$(\tau \to \infty, \ \alpha \to 0) \tag{4.9a}$$

$$\sim \tfrac{1}{2}\alpha^{-2} \exp\{-\tfrac{1}{2}(\tau/\alpha)\}\{1 + O(\alpha^{-3})\} \quad (\tau \to \infty, \ \alpha \to \infty). \tag{4.9b}$$

Invoking these approximations in (4.4), we place the result in the form

$$\zeta \sim \zeta_g + \zeta_d + \zeta_c \quad (\tau \to \infty), \tag{4.10}$$

where:
$$\zeta_g = h \int_0^\infty J_0(\alpha\eta) \exp\{-2\alpha^2\tau\} \sin(\alpha^{\frac{1}{2}}\tau)\alpha^{\frac{1}{2}} d\alpha \tag{4.11}$$

is the asymptotic approximation to the contributions of the complex-conjugate poles and represents a damped gravity wave;

$$\zeta_d = -2h(\pi\tau^3)^{-\frac{1}{2}} \int_0^\infty J_0(\alpha\eta) \exp\{-\alpha^2\tau\}\alpha^3 d\alpha \tag{4.12}$$

† The subsequent, asymptotic development makes free use of the methods described by Copson (1965).

is the asymptotic approximation to the contribution of the continuous spectrum of the cut, $\sigma = [-\infty, -\alpha^2]$, and represents an essentially diffusive disturbance; and

$$\zeta_c = \tfrac{1}{2}h\int_0^\infty J_0(\alpha\eta)\exp\{-\tfrac{1}{2}(\tau/\alpha)\}d\alpha \tag{4.13}$$

is the asymptotic approximation to the contributions of the negative-real poles and represents a creep wave. We observe that our definition of a damped gravity wave is precise by virtue of the requirement that $\sigma_\pm$ be complex. The distinction between diffusive and creeping disturbances is sharp only for $\tau \to \infty$; in particular, the disturbance associated with σ_- in $\alpha > \alpha_c$ is distinctly diffusive as $\alpha \to \infty$ (in which limit it is also negligible), but is essentially identical with that associated with σ_+ as $\alpha \to \alpha_c +$.

Carrying out the stationary-phase approximation to (4.11), which is equivalent to a saddle-point approximation to the oscillatory component, $0 < \alpha < \alpha_c$, of (4.4), we obtain

$$\zeta_g \sim 2^{-\frac{5}{2}}h\tau^3\eta^{-4}\exp(-\tfrac{1}{8}\tau^5\eta^{-4})\sin(\tfrac{1}{4}\tau^2\eta^{-1}) \quad (1 \ll \tau \ll \eta \ll \tau^2) \tag{4.14a}$$

$$= 2^{\frac{1}{2}}h\eta^{-\frac{5}{2}}\omega^{\frac{3}{2}}\exp\{-2(\tau/\eta^2)\omega^2\}\sin\omega, \tag{4.14b}$$

where ω is the similarity variable of (1.5*b*).

Invoking the Hankel-transform pairs §8.3 (5) and §8.2 (53) in the tables of Erdélyi, Magnus, Oberhettinger & Tricomi (1953), we evaluate the integrals in (4.12) and (4.13) as follows:

$$\zeta_d = 2h(\pi\tau^3)^{-\frac{1}{2}}(\partial/\partial\tau)\int_0^\infty J_0(\alpha\eta)\exp\{-\alpha^2\tau\}\alpha\,d\alpha \tag{4.15a}$$

$$= h(\pi\tau^7)^{-\frac{1}{2}}\{1-(\eta^2/4\tau)\}\exp(-\eta^2/4\tau) \tag{4.15b}$$

and

$$\zeta_c = 2h(\partial/\partial\tau)\int_0^\infty J_0(\alpha\eta)\exp(-\tau/4\alpha)\sinh(\tau/4\alpha)\,\alpha\,d\alpha \tag{4.16a}$$

$$= h\eta^{-1}(\partial/\partial\tau)\{\tau J_1([\tau\eta]^{\frac{1}{2}})K_1([\tau\eta])^{\frac{1}{2}}\} \tag{4.16b}$$

$$= h\eta^{-1}\gamma\{J_0(\gamma)K_1(\gamma) - J_1(\gamma)K_0(\gamma)\} \tag{4.16c}$$

$$\sim 2^{\frac{1}{2}}h\,e^{-\gamma}\cos\gamma\{1+O(\gamma^{-1})\}, \tag{4.16d}$$

where

$$\gamma = (\tau\eta)^{\frac{1}{2}} = (grt/\nu)^{\frac{1}{2}}. \tag{4.17}$$

The disturbance described by (4.14) tends to the classical, Cauchy–Poisson solution of the point-impulse problem (Lamb 1932, §255) as $\eta^2/\tau \to \infty$ with ω fixed, in which limit it appears as a true similarity solution in the variable ω and represents a balance between gravitational and inertial forces. It dominates both ζ_d and ζ_c in $1 \ll \tau \ll \eta \ll \tau^2$ but ultimately decays quite rapidly as $\tau \to \infty$ with η fixed. It is plotted in figure 2 for η in the neighbourhood of the maximum of the envelope (the coefficient of $\sin\omega$), namely

$$2^{-\frac{5}{2}}h\tau^3\eta^{-4}\exp(-\tfrac{1}{8}\tau^5\eta^{-4}) = 0{\cdot}518h/\tau^2 \quad \text{at} \quad \eta = \eta_m = (\tfrac{1}{8}\tau^5)^{\frac{1}{4}}. \tag{4.18}$$

The accuracy of the asymptotic approximation may be poor for $\eta < \eta_m$ [in consequence of the approximation (4.8*a*)], but we expect it to be qualitatively valid.

The disturbance described by (4.15) is a similarity solution in the variable η^2/τ and represents the diffusion of vorticity under the action of viscous and inertial forces. It is negligible for $\tau \gg 1$, even though it dominates both ζ_g and ζ_c if $\tau^{-1} \ll \eta \ll \tau^{\frac{1}{2}}$.

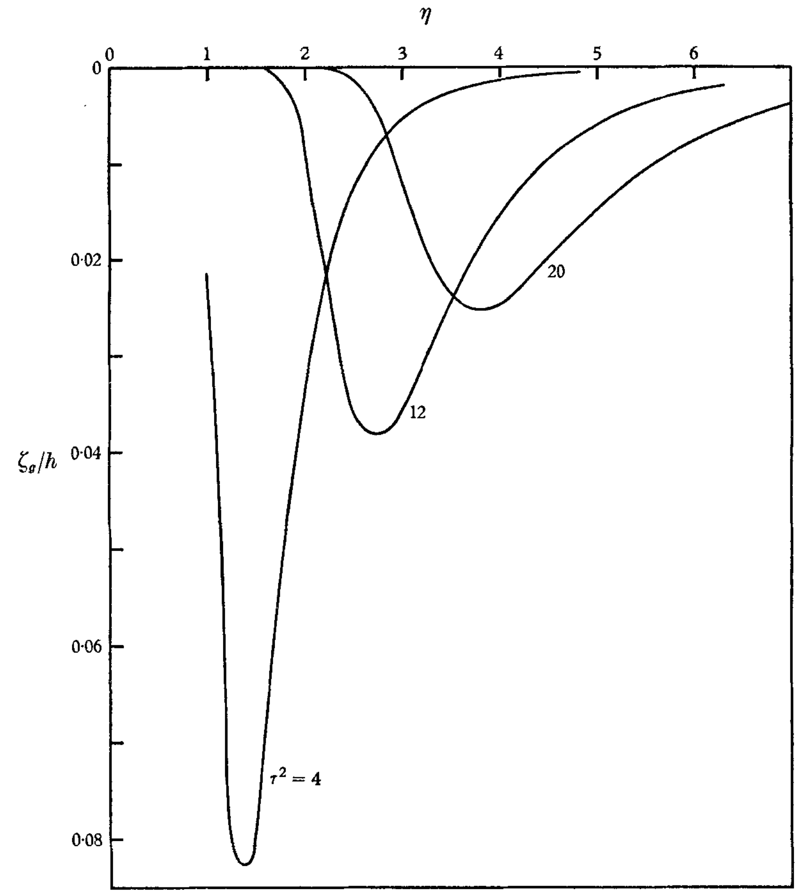

FIGURE 2. The asymptotic form of the gravity wave generated by a point impulse, as given by (4.14) for $\tau^2 = 4$, 12 and 20; ζ is positive downwards.

The disturbance described by (4.16) is a similarity solution in the variable γ and represents a balance between gravitational and viscous forces. It dominates both ζ_g and ζ_d if $\eta = O(\tau^{-1})$ and is especially interesting for a very viscous fluid. It is oscillatory in character, but the resulting wave pattern is damped at the asymptotic rate of $1/e$ per cycle. It is plotted as a function of the similarity variable γ in figure 3*a* and as a function of $\tau\eta$ in figure 3*b*. The latter plot gives a linearly scaled representation of the subsiding cavity, although it must be recalled that the asymptotic approximation (4.13) is not uniformly valid as $\eta \to 0$.

We conclude that the free-surface response of a viscous liquid to an impulse is asymptotically separated into three, distinct zones:

(i) a gravity-wave zone, $\tau \ll \eta \ll \tau^2$, in which $\zeta \sim \zeta_g = O(h\tau^3/\eta^4)$;

(ii) a null zone, $\tau^{-1} \ll \eta \ll \tau$, in which each of ζ_g, ζ_d and ζ_c is small in consequence of the joint action of diffusion and dispersion;

(iii) a creep-wave zone, $\eta = O(\tau^{-1})$, in which $\zeta \sim \zeta_c = O(h/\eta)$.

The scales of ζ_c and ζ_g as functions of η for fixed $\tau \gg 1$ are so disparate that it is not feasible to compare them in a common plot.

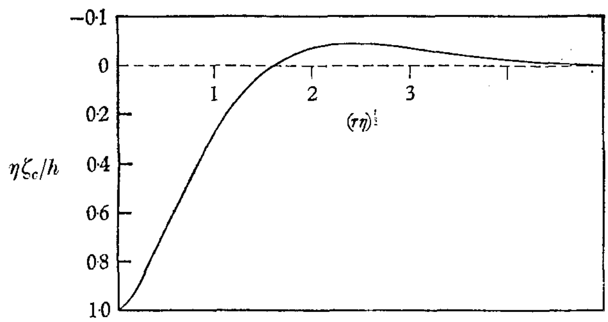

FIGURE 3*a*. The asymptotic form of the creep wave generated by a point impulse, as given by the similarity solution (4.16); ζ is positive downwards.

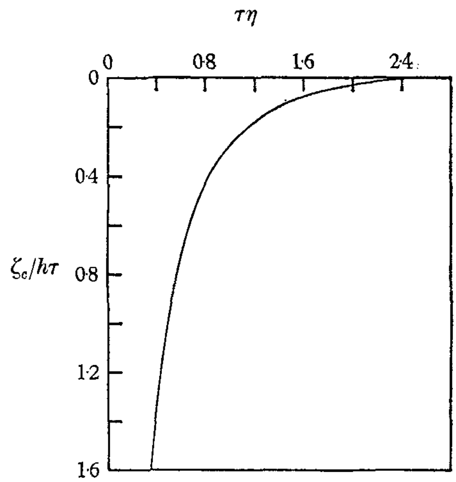

FIGURE 3*b*. The asymptotic form of the creep wave generated by a point impulse, as given by the similarity solution (4.16); ζ is positive downwards.

5. Initial-cavity problem

We now consider the disturbance produced by the initial displacement (see figure 4)

$$\zeta_0(r) = d\exp\{-(r/a)^2\}\{1-(r/a)^2\}, \tag{5.1}$$

which represents a cavity with a lip such that the volumetric constraint of (2.8) is satisfied and

$$E_0 = \tfrac{1}{8}\pi\rho g d^2 a^2. \tag{5.2}$$

Substituting (5.1) into (3.2) and invoking §8.3(5) in Erdélyi *et al.* (1953), we obtain

$$Z_0(k) = \tfrac{1}{8}da^2(ka)^2\exp\{-\tfrac{1}{4}(ka)^2\} \quad (\Phi_0 = 0). \tag{5.3}$$

Substituting (5.3) into (3.11) and (3.13) and introducing the dimensionless quantities of (4.2), (4.3) and (1.3), we obtain

$$\zeta(r,t) = 2\lambda^2 d\int_0^\infty J_0(\alpha\eta)\exp\{-\gamma\alpha^2\}\chi_1(\tau,\alpha)\alpha^3 d\alpha, \tag{5.4}$$

where

$$\chi_1(\tau,\alpha) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left\{1-\frac{\alpha}{\Delta(\sigma,\alpha)}\right\}e^{\sigma\tau}\frac{d\sigma}{\sigma}. \tag{5.5}$$

Comparing (5.5) with (4.5) and invoking (4.9), we obtain

$$\chi_1(\tau,\alpha) = \alpha\int_\tau^\infty \chi(\eta,\alpha)\,d\eta \tag{5.6a}$$

$$\sim \exp\{-2\alpha^2\tau\}\cos(\alpha^{\frac{1}{2}}\tau)\{1+O(\alpha^{\frac{9}{4}})\}$$
$$-2(\pi\tau^3)^{-\frac{1}{2}}\exp\{-\alpha^2\tau\}\{1+O(\alpha/\tau)\} \quad (\tau\to\infty,\alpha\to 0) \tag{5.6b}$$

$$\sim \exp\{-\tfrac{1}{2}(\tau/\alpha)\}\{1+O(\alpha^{-3})\} \quad (\tau\to\infty,\alpha\to\infty). \tag{5.6c}$$

Substituting (5.6b, c) into (5.4), letting $\tau \to \infty$, and decomposing the asymptotic representation as in (4.10), we obtain

$$\zeta_g = 2\lambda^2 d \int_0^\infty J_0(\alpha\eta) \exp\{-(\lambda+2\tau)\alpha^2\} \cos(\alpha^{\frac{1}{2}}\tau)\alpha^3 d\alpha, \tag{5.7}$$

$$\zeta_d = -2\lambda^2 d(\pi\tau^3)^{-\frac{1}{2}} \int_0^\infty J_0(\alpha\eta) \exp\{-(\lambda+\tau)\alpha^2\}\alpha^3 d\alpha, \tag{5.8}$$

and

$$\zeta_c = 2\lambda^2 d \int_0^\infty J_0(\alpha\eta) \exp\{-\lambda\alpha^2 - \tfrac{1}{2}(\tau/\alpha)\}\alpha^3 d\alpha. \tag{5.9}$$

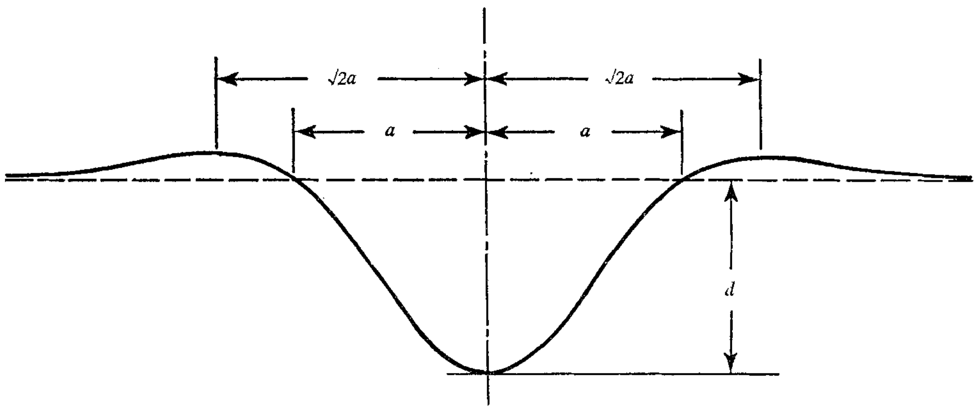

FIGURE 4. The cavity and lip described by (5.1). The volumetric displacement in $r > a$ (the lip) is equal and opposite to that in $r < a$ (the cavity).

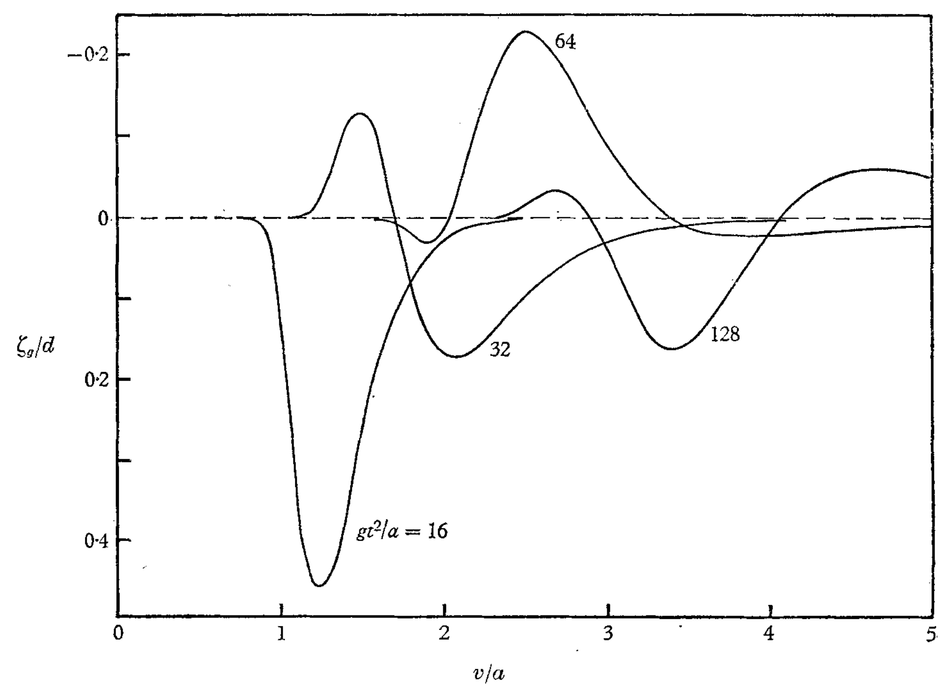

FIGURE 5. The asymptotic response of an inviscid liquid to the initial displacement of figure 4; ζ is positive downwards.

Evaluating the integrals of (5.7) and (5.8) as in §4 and approximating that in (5.9) by Laplace's method, we obtain

$$\zeta_g \sim -2^{-\frac{9}{2}}\lambda^2 d\tau^6\eta^{-7}\exp\{-(\lambda+2\tau)(\tau/2\eta)^4\}\cos(\tfrac{1}{4}\tau^2\eta^{-1}) \quad (1 \ll \tau \ll \eta \ll \tau^2), \tag{5.10}$$

$$\zeta_d \sim -\lambda^2 d(\lambda+\tau)^{-2}(\pi\tau^3)^{-\frac{1}{2}}\{1-\tfrac{1}{4}(\lambda+\tau)^{-1}\eta^2\}\exp\{-\tfrac{1}{4}(\lambda+\tau)^{-1}\eta^2\}, \tag{5.11}$$

and

$$\zeta_c \sim \tfrac{1}{2}(\tfrac{1}{3}\pi)^{\frac{1}{2}}d\lambda^{\frac{1}{2}}\tau J_0\{\eta(\tau/4\lambda)^{\frac{1}{3}}\}\exp\{-\tfrac{3}{2}(\tfrac{1}{2}\lambda\tau^2)^{\frac{1}{3}}\}. \tag{5.12}$$

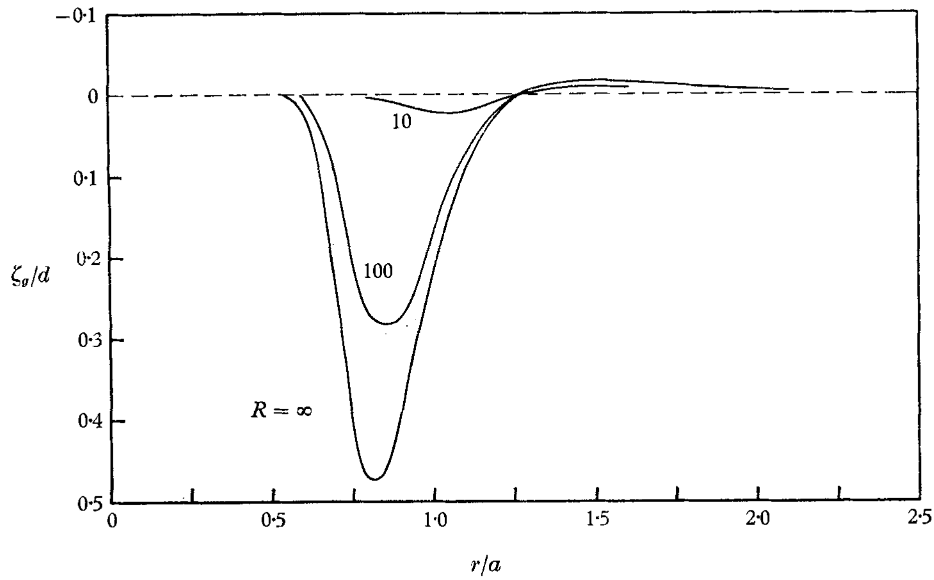

FIGURE 6*a*. The asymptotic response of a viscous liquid to the initial displacement of figure 4 for $gt^2/a = 8$ and $R = 10$, 100 and ∞; ζ is positive downwards.

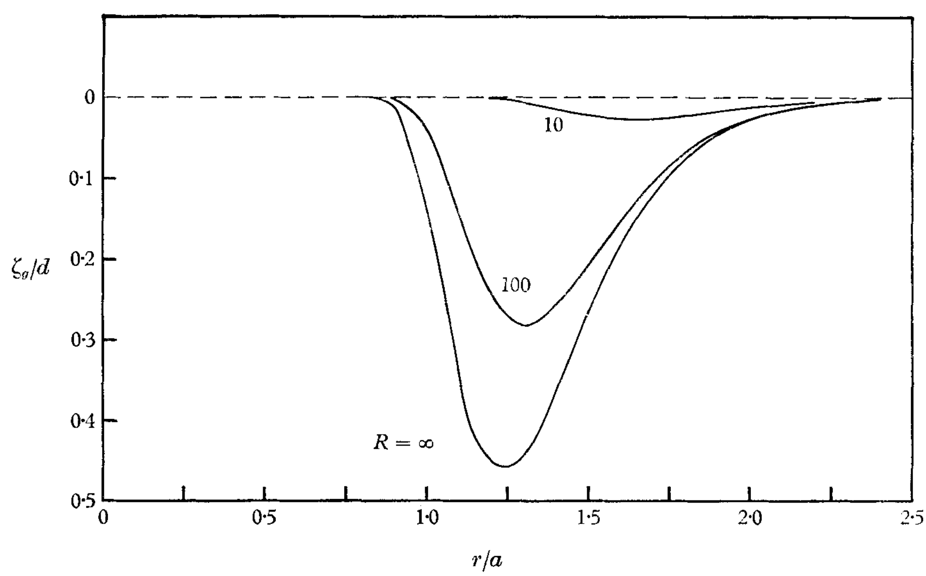

FIGURE 6*b*. The asymptotic response of an inviscid liquid to the initial displacement of figure 4 for $gt^2/a = 16$ and $R = 10$, 100, and ∞; ζ is positive downwards.

The approximations (5.11) and (5.12) are uniformly valid with respect to η for fixed λ, but (5.12) is not uniformly valid as $\lambda \to 0$. We again may identify ζ_g, ζ_d and ζ_c as a damped gravity wave, an essentially diffusive disturbance, and a creep wave, respectively, but only (5.11) as a similarity solution. Both ζ_d and ζ_c are asymptotically negligible for all η as $\tau \to \infty$ with λ fixed, and there is no counterpart of the creep-wave zone of the point-impulse problem unless $\lambda \ll 1$. The results for $\lambda \ll 1$ are similar to those for the point-impulse problem.

Restoring the original variables in (5.10), we place the result in the form

$$\zeta_g/d \sim -2^{-\frac{5}{2}}\omega_a^3 \xi^{-7} \exp\{-(\tfrac{1}{4}\omega_a^2 + 4R^{-1}\omega_a^{\frac{5}{2}})\xi^{-4}\}\cos(\omega_a/\xi) \quad (2R^{-\frac{1}{3}}\omega_a \ll \xi \ll 4\omega_a^2), \tag{5.13}$$

where

$$\omega_a = gt^2/4a, \quad \xi = r/a, \tag{5.14}$$

and R is given by (1.3). The result (5.13) is plotted in figure 5 in the inviscid limit ($R = \infty$) with ω_a as a parameter and in figures 6*a* and 6*b* with R as a parameter.

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